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# Degenerate Unsteady Compressible Flows

J. H. GIESE

DEPARTMENT OF THE ARMY PROJECT Nos. 503-03-001 AND 503-06-002
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT Nos. TB3-0108H AND TB3-0007K

BALLISTIC RESEARCH LABORATORIES

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B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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# DEGENERATE UNSTEADY COMPRESSIBLE FLOWS

## ABSTRACT

Characteristic conditions are derived for non-linearized three dimensional compressible flows in which the entropy of any fluid particle remains constant.

Irrotational isentropic flows for which the maps x, y, z, t

$$\longrightarrow \partial\phi/\partial x,\ \partial\phi/\partial y,\ \partial\phi/\partial z,\ \partial\phi/\partial t$$

have fewer than four dimensions have been studied by applying a degenerate Legendre transformation to the equation of the velocity potential function, $\phi$. The class of such flows includes steady conical flows and pseudo-stationary flows, for example. All results presented are in the nature of statements about geometrical properties of the map, of certain characteristic hypersurfaces in space-time, etc.

A number of problems which could be expected to lead to pseudo-stationary flows are outlined. Most, but not all, are well known problems that have been solved at least in linearized form.

## 1. Introduction.

The development of the theory of compressible perfect fluids has been impeded by the fact that the basic equations are non-linear. Therefore it has been profitable to consider special examples to discover general principles and to gain insight into the structure of flows. Specializations are usually based on such properties as irrotationality, reduction in number of dimensions, radial, cylindrical, or axial symmetry, etc.

For steady irrotational flow it has been found instructive to study those flows for which the mapping $(x, y, z) \rightarrow (u, v, w) = \nabla \phi$, where $\phi$ is the velocity potential, takes the physical space onto a one-or two-dimensional image in the hodograph space [5] [1]. This leads to consideration of the class of conical flows which are interesting and important not only in their own right, but also for the fact that they appear to be connected with the singularities of steady flow fields near the tips or edges of bodies with discontinuous normals. That is, the limits attained by the velocity components and other flow functions at the tip will depend, in general, on the direction from which the tip is approached. An argument can be devised which makes it plausible to conclude that the $(u, v, w)$ image or hodograph is the same as that for the flow which would occur about the tangent cone at the tip under the same free stream conditions.

Such considerations have suggested that it would be equally profitable to undertake the study of non-steady irrotational flows for which the mapping $(t, x, y, z) \rightarrow (\partial\phi/\partial t, u, v, w) = (\partial\phi/\partial t, \nabla\phi)$, where $\phi$ is the velocity potential, takes space-time onto a one-, two-, or three-dimensional image. The image will be called the hodograph, and when it has fewer than four dimensions, the hodograph and the flow will be said to be degenerate. The study leads, eventually, to the consideration of non-steady conical flows, also referred to as pseudo-stationary or self-similar flows. Such flows occur in the establishment of steady conical flows about conical bodies in shock tubes, not to mention other examples discussed at the end of the paper. It is to be expected that they are also related to the behavior of flow fields near singular points in space-time. That is, the limits attained by velocity components and other flow functions at the singular point should depend, in general, on the direction from which the singular point is approached. If this is in a region of potential flow, one would conjecture that the $(\partial\phi/\partial t, u, v, w)$ image or hodograph is the same as that for one of the degenerate unsteady flows considered here. This phenomenon would occur, for example, in the one-dimensional flow behind a piston which is withdrawn with increasing speed and which experiences a discontinuous increase in speed at some instant. The singular point would be on the piston curve at the time and place where the discontinuity in velocity occurs.

1. Bracketed numbers designate references at the end of the paper.

Much of the discussion is concerned with charachteristic subspaces of space-time or of the hodographs. To clarify this discussion, and also because it casts some light on the general wave nature of unsteady flows, a section on the characteristic conditions for general unsteady flows has been included.

## 2. Fundamental Equations.

Non-steady, non-viscous flow is governed by the equations of motion

$$(2.1) \qquad \partial u_i/\partial t + u_j \partial u_i/\partial u_i/\partial x_j = - \rho^{-1} \partial p/\partial x_i$$

the equation of continuity

$$(2.2) \qquad \partial \rho/\partial t + \partial (\rho u_j)/\partial x_j = 0$$

a pressure density relation

$$(2.3) \qquad p = e^{s/c_v} \rho^\gamma$$

and constancy of entropy for a fluid particle (between passages through successive shock fronts)

$$(2.4) \qquad \partial s/\partial t + u_j \partial s/\partial x_j = 0$$

Here $u_i(x, t)$ $(i = 1, 2, 3)$ are cartesian velocity components at time $t$ at the point with rectangular coordinates $x_1$, $x_2$, $x_3$; $p(x, t)$, $\rho(x, t)$, and $s(x, t)$ are the pressure, density, and entropy per unit mass, respectively; and $\gamma$ is the ratio of the specific heat at constant pressure to that at constant volume. The convention has been adopted that repeated indices in any sum and imply summation over their range. The pressure can be eliminated from (2.1) by means of (2.3). It will be assumed hereafter that this has been done.

Occasionally it will be more advantageous to use vector notation. Then the preceding equations take the form

$$(2.1v) \qquad \partial \overline{u}/\partial t + \overline{u} \cdot \nabla \overline{u} = - \rho^{-1} \nabla_p$$

$$(2.2v) \qquad \partial \rho/\partial t + \nabla \cdot (\rho \overline{u}) = 0$$

$$(2.4v) \qquad \partial s/\partial t + \overline{u} \cdot \nabla s = 0$$

In some parts of the following discussion it will be convenient to consider irrotational isentropic flows. For these $s$ = constant, and

$$(2.5) \qquad \partial u_i/\partial x_j = \partial u_j/\partial x_i$$

Now there exists a velocity potential function $\phi$ such that

$$(2.6) \qquad u_i = \partial \phi/\partial x_i$$

Then (2.1), (2.3), and (2.6) imply Bernoulli's equation $\partial\phi/\partial t + \frac{1}{2} u_i u_i + \gamma\, p/(\gamma-1)\rho =$ $df/dt$ for some function $f(t)$. Since $f(t)$ can be absorbed into $\phi$ without affecting (2.1) to (2.6), Bernoulli's equation can be expressed as

$$(2.7) \qquad a^2 = \partial p/\partial\rho = \gamma p/\rho = -(\gamma - 1)(\tfrac{1}{2} u_i u_i + \partial\phi/\partial t)$$

where $a$ is the speed of sound. Finally, eliminate $p$ and $\rho$ from (2.1) and (2.2) by means of (2.3) and (2.7) to obtain

$$(2.8) \qquad (a^2\delta_{ij} - u_i u_j)\, \partial\, \phi/\partial x_i \partial x_j - 2u_j \partial^2\phi/\partial x_j \partial t - \partial^2\phi/t^2 = 0$$

where Kronecker's delta, $\delta_{ij} = 1\ (0)$ accordingly as $i = (\neq)\ j$.

## 3. Characteristic Conditions.

A standard exercise in the theory of partial differential equations is Cauchy's problem. For the system (2.1v), (2.2v), and (2.4v) this can be stated as follows. Consider a three-dimensional initial hypersurface in space-time.

$$(3.1) \qquad \bar{x} = \bar{X}(r_1, r_2, r_3) = \bar{X}(r), \qquad t = T(r_1, r_2, r_3) = T(r)$$

where the

$$(3.2) \qquad \text{rank of } \| \partial\bar{X}/\partial r_i, \quad \partial T/\partial r_i \| = 3$$

Construct solutions $\bar{u}(\bar{x}, t)$, $\rho(\bar{x}, t)$, and $s(\bar{x}, t)$ which assume on (3.1) the prescribed initial values

$$(3.3) \qquad \begin{aligned} \bar{u}(\bar{X}(r),\ T(r)) &= \bar{U}(r) \\ \rho(\bar{X}(r),\ T(r)) &= R(r) \\ s(\bar{X}(r),\ T(r)) &= S(r) \end{aligned}$$

subject to appropriate assumptions about the continuity and differentiability of the functions (3.1) and (3.3).

If the function $T(r)$ is a constant, $T_o$, then (3.1) is merely some three-dimensional region of the $x_1x_2x_3$ space at a fixed time $t = T_o$. If $T(r)$ is not a constant function, then for the solutions of $T(r) = T_o$ = constant the functions $\bar{X}(r)$ define a surface in the $x_1x_2x_3$-space,

the shape, location, and orientation of which depend on the time $T_o$. In other words, (3.1) can be interpreted as defining a moving twisting, deforming surface in the $x_1x_2x_3$-space.

If $\bar{X}$, T, $\bar{U}$, R, and S are analytic in $r_1$, $r_2$, and $r_3$, try to expand $\bar{u}(\bar{x}, t)$, $\rho(\bar{x}, t)$, and $s(\bar{x}, t)$ in Taylor's series about some point $\bar{X}*$, T* of (3.1). The values of

(3.4) $\nabla \bar{u}$, $\partial\bar{u}/\partial t$, $\nabla\rho$, $\partial\rho/\partial t$, $\nabla s$, $\partial s/\partial t$

required for this expansion must satisfy (2.1v), (2.2v), (2.4v), and the strip conditions:

(3.5) $$\bar{X}_i \cdot \nabla\bar{u} + T_i \partial\bar{u}/\partial t = \bar{U}_i$$

(3.6) $$\bar{X}_i \cdot \nabla\rho + T_i \partial\rho/\partial t = R_i$$

(3.7) $$\bar{X}_i \cdot \nabla s + T_i \partial s/\partial t = S_i$$

where $f_i = \partial f/\partial r_i$ for any f(r), and i = 1, 2, 3. If these twenty equations have a unique solution (3.4) on (3.1), then the values of all partial derivatives of all orders are uniquely determined on (3.1). By the Cauchy-Kowalewski theorem, the Taylor's series for $\bar{u}(\bar{x}, t)$, $\rho$ ($\bar{x}$, t), and s(x, t) are unique, converge near $\bar{X}*$, T*, and satisfy (2.1v), (2.2v), (2.4v), and (3.3).

In all flow problems $\vec{u}$, $\rho$, and s must be continuous in regions of space-time between shock fronts or contact discontinuities, but (3.4) or the partial derivatives of higher order merely need to be continuous almost everywhere, so $\bar{u}$, $\rho$ and s are not necessarily analytic. Hence some partial derivatives of $\bar{u}$, $\rho$, or s may have a discontinuity. If this is interpreted as a weak disturbance, it is important from physical and computational considerations to determine how this is propagated. For simplicity, suppose some of (3.4) are discontinuous at $\bar{X}*$, T* on (3.1), though the discussion can also be adapted to partial derivatives of higher order. Then it must be impossible to determine (3.4) uniquely at $\bar{X}*$, T* from (2.1v), (2.2v), (2.4v), and (3.5) to (3.7), but this system must still be compatible. For theoretical and numerical applications the most interesting situation is that in which (3.4) fail to be uniquely determined everywhere on (3.1) rather than just at $\bar{X}*$, T*. In this case (3.1) will be said to be a <u>characteristic hypersurface</u>, and the compatibility conditions for the system of twenty linear equations, which are expressed solely in terms of (hyper-)surface data (3.1) and (3.3) and their surface (tangential) derivatives, will be called <u>characteristic conditions</u>.

An application of Cramer's rule to (3.4) yields $(\overline{X}_1\ \overline{X}_2\ \overline{X}_3)\nabla\overline{u} =$
$\sum_j(\overline{X}_{j+1}\times\overline{X}_{j+2})(\overline{U}_j - T_j\partial\overline{u}/\partial t)$, where $(\overline{A}\ \overline{B}\ \overline{C}) = \overline{A}\cdot\overline{B}\times\overline{C}$ for any vectors $\overline{A}$, $\overline{B}$, $\overline{C}$, and $j+1$ and $j+2$ are to be reduced mod 3. Now the four components $N_o$, $\overline{N}$ of a normal to (3.1) satisfy

$$(3.8)\qquad N_oT_i + \overline{N}\cdot\overline{X}_i = 0$$

By (3.2) this system has one linearly independent solution $N_o = (\overline{X}_1\ \overline{X}_2\ \overline{X}_3)$, $\overline{N} = -\sum_j T_j\overline{X}_{j+1}\times\overline{X}_{j+2}$. Hence

$$\begin{aligned}
&N_o\nabla\overline{u} = \overline{N}\partial\overline{u}/\partial t + \textstyle\sum_j\overline{X}_{j+1}\times\overline{X}_{j+2}\overline{U}_j\\
(3.9)\qquad &N_o\nabla\rho = \overline{N}\partial\rho/\partial t + \textstyle\sum_j\overline{X}_j+1\times\overline{X}_{j+2}R_j\\
&N_o\nabla s = \overline{N}\partial s/\partial t + \textstyle\sum_j X_{j+1}\times\overline{X}_{j+2}S_j
\end{aligned}$$

Then by (2.1v), (2.2v), (2.4v), and (3.9)

$$(3.10)\qquad \begin{aligned}&(N_o + \overline{U}\cdot\overline{N})\partial\overline{u}/\partial t + R^{-1}\,\overline{N}\left[A^2\partial\rho/\partial t + (\partial P/\partial S)\partial s/\partial t\right] +\\ &\textstyle\sum_j(\overline{X}_{j+1}\times\overline{X}_{j+2})\left[\bullet\overline{U}\overline{U}_j + R^{-1}\left\langle A^2R_j + S_j\partial P/\partial S\right\rangle\right] = 0\end{aligned}$$

$$(3.11)\qquad (N_o + \overline{U}\cdot\overline{N})\partial\rho/\partial\ t + R\overline{N}\bullet\partial\overline{u}\ t + \textstyle\sum_j(\overline{X}_{j+1}\times\overline{X}_{j+2})\bullet(R\overline{U})_j = 0$$

$$(3.12)\qquad (N_o + \overline{U}\cdot\overline{N})\partial s/\partial\ t + \textstyle\sum_j(\overline{U}\ \overline{X}_{j+1}\overline{X}_{j+2})S_j = 0$$

where $P = e^{S/c_v}R^{\gamma}$ and $A^2 = \partial P/\partial R$. The scalar product of $\overline{N}$ and (3.10) yields

$$\begin{aligned}&(N_o + \overline{U}\cdot\overline{N})N\cdot\partial\overline{u}/\partial t + R^{-1}N^2\left[A^2\partial\rho/\partial t + (\partial P/\partial S)\partial s/\partial t\right] +\\ &\textstyle\sum_j\left[\overline{N}\cdot\overline{U}_j(\overline{U}\ \overline{X}_{j+1}\ \overline{X}_{j+2}) + R^{-1}(A^2R_j + S_j\partial P/\partial S)(\overline{N}\ \overline{X}_{j+1}\ \overline{X}_{j+2})\right] = 0\end{aligned}$$

The determinant of the coeficients of

$$(3.14)\qquad \overline{N}\cdot\partial\overline{u}/\partial t,\qquad \partial\rho/\partial t,\qquad \partial s/\partial t$$

in (3.11) to (3.13) is

$$(3.15)\qquad \Delta = (N_o + \overline{U}\cdot\overline{N})\left[(N_o + \overline{U}\cdot\overline{N})^2 - A^2N^2\right]$$

If $\Delta \neq 0$, then (3.14) are uniquely determined, and since $N_o + \bar{U}\cdot\bar{N} \neq 0$, $\partial\bar{u}/\partial t$ are unique by (3.10) If in addition $N_o \neq 0$, then by (3.9) $\nabla\bar{u}$, $\nabla\rho$, and $\nabla s$ are unique. Thus a necessary condition for non-unique values of (3.4) is

$$(3.16) \qquad N_o \Delta = 0$$

Case 1: If the Jacobian

$$(3.17) \qquad N_o = (\bar{X}_1\ \bar{X}_2\ \bar{X}_3) = 0$$

its rank is two by (3.2), and also $T_i \neq 0$ for some i. Hence $F(\bar{X}) = 0$ for some F. In other words, <u>the initial surface in the $x_1x_2x_3$ - space does not vary with the passage of time.</u> Now (3.9) are compatible since $\bar{N}\cdot\bar{X}_i = 0$, and hence $\bar{N}\times(\bar{X}_{j+1}\times\bar{X}_{j+2}) = 0$. Their unique solutions satisfy

$$N^2\partial\bar{u}/\partial t = \sum_j(\bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2})\bar{U}_j, \qquad N^2\partial\rho/\partial t = \sum_j(\bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2})R_j$$

$$N^2\partial s/\partial t = \sum_j \bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2})S_j$$

By means of these equations eliminate $\partial\bar{u}/\partial t$, $\partial\rho/\partial t$, and $\partial s/\partial t$ from (3.10) to (3.12) to obtain

$$(3.18) \qquad \sum_j(\bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2})\left[\bar{U}\cdot\bar{N}\bar{U}_j + R^{-1}\bar{N}(A^2R_j + S_j\partial P/\partial S)\right] + N^2\sum_j(\bar{X}_{j+1}\times\bar{X}_{j+2})\left[\cdot\bar{U}\bar{U}_j + R^{-1}(A^2R_j + S_j\partial P/\partial S)\right] = 0$$

$$(3.19) \qquad \sum_j\left[(\bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2})N + N^2\bar{X}_{j+1}\times\bar{X}_{j+2}\right]\cdot\ (R\bar{U})_j = 0$$

$$(3.20) \qquad \sum_j\left[\bar{U}\cdot\bar{N}(\bar{N}\ \bar{X}_{j+1}\ \bar{X}_{j+2}) + N^2(\bar{U}\ \bar{X}_{j+1}\ \ \bar{X}_{j+2}\ )\right]S_j = 0$$

Case 2: Suppose $N_o \neq 0$, but

$$(3.21) \qquad N_o + \bar{U}\cdot\bar{N} = 0$$

Then the compatibility conditions for (3.10) become

$$(3.22) \qquad \bar{N}\times\sum_j(\bar{X}_{j+1}\times\bar{X}_{j+2})\left[\cdot\bar{U}\bar{U}_j + R^{-1}(A^2R_j + S_j\partial P/\partial S)\right] = 0$$

If these are satisfied, then (3.10)is equivalent to the single equation (3.13), which now takes the form

$$(3.13.2) \quad 0 = R^{-1}N^2 \; A^2\partial\rho/\partial t + (\partial P/\partial S)\partial s/\partial t +$$

$$_j \; \overline{N}\cdot\overline{U}_j \; (\overline{U}\,\overline{X}_{j+1}\,\overline{X}_{j+2}) + R^{-1}(A^2R_j + S_j\partial P/\partial S)(\overline{N}\,\overline{X}_{j+1}\,\overline{X}_{j+2})$$

while (3.11) becomes

$$(3.11.2) \quad R\overline{N}\cdot\partial\overline{u}/\partial t + {}_j(\overline{X}_{j+1} \;\; \overline{X}_{j+2})\cdot(R\overline{U})_j = 0$$

Finally (3.12) yields

$$(3.23) \quad {}_j(\overline{U}\,\overline{X}_{j+1} \;\; \overline{X}_{j+2})S_j = 0$$

By (3.11.2) and (3.13.2) three of the five functions (3.14) are arbitrary. Hence there is a three-parameter family of solutions (3.4).

To interpret (3.21) proceed as follows. Suppose a solution

$u(\overline{x}, t)$, $\rho(\overline{x}, t)$, $s(\overline{x}, t)$ of (2.1v), (2.2v), and (2.4v) is known. First seek an integral $t = f(\overline{x})$ of (3.21). In this case $N_o = -1$, $\overline{N} = \nabla f \equiv \overline{p}$, and (3.21 becomes

$$(3.24) \quad \overline{u}\cdot\overline{p} - 1 = 0$$

where $\overline{u} = \overline{u}(\overline{x}, f)$. Solutions of this first order partial differential equation can be constructed by passing through each point of some two-dimensional initial manifold

$$(3.25) \quad \overline{x} = \overline{x}^*(r_1, r_2), \qquad t = t^*(r_1, r_2)$$

integral curves of the system

$$d\overline{x}/dr_3 = \overline{u}, \qquad dt/dr_3 = \overline{u}\cdot\overline{p} = 1$$

$$(3.26) \quad d\overline{p}/dr_3 = - \; ( \;\; \overline{u} \cdot\overline{p} - (p\cdot\partial\overline{u}/\partial t)\overline{p}$$

with initial conditions $\overline{u} = \overline{u}^*(r_1, r_2)$(known, of course) and $\overline{p} = p^*(r_1, r_2)$ chosen to be solutions of $\overline{u}^* \cdot\overline{p}^* = 1$ and $\partial t^*/\partial r_\alpha + \overline{p}^* \cdot\partial\overline{x}^*/\partial r_\alpha = 0$ $(\alpha = 1, 2)$. By (3.26) assume $r_3 = t$ with no loss of generality. Hence $d\overline{x}/dt = \overline{u}$. In other words, at all times t the surface $x = \overline{x}(r_1, r_2; t)$ is always composed of the same particles, i.e., the integral surfaces $t=f(x)$ of (3.21) <u>are particle fronts.</u>

Integral surfaces of (3.21) which cannot be represented by $t = f(\overline{x})$ must satisfy some equation $g(\overline{x}) = 0$, where g is independent of t. On such a surface $N_o = 0$, contrary to the hypotheses for this case.

Case 3: Suppose $N_o(N_o + \overline{U}\cdot\overline{N}) \neq 0$, but

(3.27) $$(N_o + \overline{U}\cdot\overline{N})^2 - A^2N^2 = 0$$

By (3.12) $\partial s/\partial t$ is unique. A compatibility condition for (3.11) and (3.13) is

$$\sum_j \Big\{ (\overline{U}\ \overline{X}_{j+1}\ \overline{X}_{j+2}) \left[ (N_o + \overline{U}\cdot\overline{N})\overline{N}\cdot\overline{U}_j - R^{-1}N^2(A^2R_j + S_j\partial P/\partial S) \right] +$$

(3.28) $$(\overline{N}\ \overline{X}_{j+1}\ \overline{X}_{j+2})\ (N_o + \overline{U}\cdot\overline{N})R^{-1}(A^2R_j + S_j\partial P/\partial S) -$$
$$A^2N^2(\overline{U}_j\ \overline{X}_{j+1}\ \overline{X}_{j+2}) \Big\} = 0$$

If this is satisfied, only three of (3.10) and (3.11) are independent, and there is a one-parameter family of solutions (3.4).

To interpret (3.27), again suppose a solution of $\overline{u}(\overline{x}, t)$, $\rho(\overline{x}, t)$, $s(\overline{x}, t)$ of (2.1v), (2.2v), and (2.4v) is known. Seek integrals $t = f(\overline{x})$ of (3.27). Again $N_o = -1$, $\overline{N} = \nabla f \equiv \overline{p}$, and (3.27) becomes

(3.29) $$(\overline{u}\cdot\overline{p} - 1)^2 - a^2 p^2 = 0$$

Solutions of this first order partial differential equation can be constructed by passing through each point of some two-dimensional initial manifold (3.25) integral curves of the system

$$d\overline{x}/dr_3 = (\overline{u}\cdot\overline{p} - 1)\overline{u} - a^2\overline{p}, \qquad dt/dr_3 = \overline{p}\cdot d\overline{x}/dr_3 = \overline{u}\cdot\overline{p} - 1$$

(3.30)

$$d\overline{p}/dr_3 = -(\overline{u}\cdot\overline{p} - 1)\ (\nabla\overline{u})\cdot\overline{p} + p^2\ a\nabla a$$
$$- \left[ (\overline{u}\cdot\overline{p} - 1)\ \overline{u}_t\cdot\overline{p} - aa_t p^2 \right] \overline{p}$$

with initial conditions $\overline{p}*(r_1, r_2)$ determined by

(3.31) $$(u*\cdot p* - 1)^2 - a^{*2}\ P*^2 = 0$$

(3.32) $$\partial t^*/\partial r_\alpha + \overline{p}*\cdot\partial\overline{x}*/\partial r_\alpha = 0, \qquad \alpha = 1, 2$$

An important special case is that for which (3.25) degenerates into a single point $\overline{x}^* =$ constant, $t^* =$ constant. Then (3.32) need not be considered, and (3.31) yields a two-parameter set of initial values $\overline{p}^*(r_1, r_2)$. The set of all integral curves of (3.30) through $\overline{x}^*$, $t^*$ for each set of initial values forms a characteristic hyperconoid. Eliminate $r_3$ from (3.30) to find

$$d\overline{x}/dt = \overline{u} - a^2\overline{p}/(\overline{u}\cdot\overline{p} - 1) \qquad (3.33)$$

and a similar equation for $d\overline{p}/dt$. At each instant $t$, the points $\overline{x}(r_1, r_2 ; t)$ on the characteristic hyperconoid integrals of (3.33) determine the boundary of the region into which a weak disturbance at (or sound wave from $\overline{x}^*$ at time $t^*$ has propagated. By (3.29) and (3.33) $(d\overline{x}/dt - \overline{u})^2 = a^2$. Thus to find the possible initial velocities $d\overline{x}/dt$ at time $t^*$, to draw all vectors from the point $\overline{x}^*$ in $x_1x_2x_3$-space to the surface of the sphere with center at $\overline{x}^* + \overline{u}^*$ and radius $a^*$. This yields the well known result that weak disturbances at $\overline{x}^*$ at time $t^*$ propagate in all directions in subsonic flows, but only into the downstream nappe of the Mach conoid with vertex at $x^*$ in supersonic flows.

To put the results just obtained into a more familiar light consider one-dimensional unsteady flow. Now (3.1) can be given the form $t = t(r_1)$, $\overline{x} = (x(r_1), r_2, r_3)$, and $\overline{u} = (u(r_1), 0, 0)$. (3.21) and (3.23) yield

$$dx = udt, \qquad ds = 0 \qquad (3.33)$$

while (3.27) and (3.28) yield $(dx - udt)^2 = a^2(dt)^2$, and if $dx \neq 0$, $(dx - udt)\, du + \rho^{-1}dt(a^2 d\rho + \partial p/\partial s\, ds) = 0$. In their usual form these are

$$dx = \pm\, adt, \qquad du = \pm\, {}^{-1}(ad\rho + a^{-1}\partial p/\partial s ds) \qquad (3.35)$$

Similarly, for

$$\overline{u} = (ux/r,\ uy/r,\ 0), \qquad r^2 = x^2 + y^2$$

$$\left[\overline{u} = (ux/r,\ uy/r,\ uz/r), \qquad r^2 = x^2 + y^2 + z^2\right]$$

where $u = u(r, t)$, and for (3.1) of the form $t = t(r_1)$,

$$\overline{x} = (r(r_1)\cos r_2,\ r(r_1)\sin r_2,\ r_3)$$

$$\left[\overline{x} = (r(r_1)\cos r_2 \cos r_3,\ r(r_1)\sin r_2 \cos r_3,\ r(r_1)\sin r_3\right]$$

equations (3.21), (3.22), (3.27), and (3.28) reduces to the characteristic equations for unsteady cylindrically [spherically] symmetrical flow.

4. **Specializations. Degenerate Legendre transformations.** (2.1), (2.2), and (2.4) form a system of non-linear partial differential equations for five functions of four independent variables. At present the theory is much more highly developed for systems with two independent variables than for three or four. Hence it is profitable to consider examples of unsteady flows with special properties that make it possible to reduce the number of independent variables. A common procedure is to consider one-dimensional or cylindrically or spherically symmetrical flows. A slightly more general technique which has proved fruitful in the study of steady flows [5] is to seek solutions of the form

$$(4.1) \qquad u_i = u_i(\mu), \ \rho = \rho(\mu), \qquad s = s(\mu)$$

where $\mu = \mu_1, \ldots, \mu_n$; $\mu_\alpha = \mu_\alpha(x_1, x_2, x_3, t) = \mu_\alpha(x, t)$; $\alpha = 1, \ldots, n$;

$1 \leqq n \leqq 4$; and the

$$(4.2) \qquad \text{rank of } \left\| \partial\mu_\alpha/\partial x_i, \ \partial\mu_\alpha/\partial t \right\| = n$$

As in the steady case, this leads to very interesting classes of flows. Several important examples of this type, investigation of which should yield valuable results both in fluid mechanics and in the theory of partial differential equations, will be mentioned in Section 8. For simplicity the discussion in this and the three following sections will be restricted to irrotational isentropic flow. Now s = constant, and

$$(4.3) \qquad u_i = \partial\phi/\partial x_i$$

where $\phi$, p, and $\rho$ satisfy the partial differential equation (2.8) and Bernoulli's equation (2.7). Let

$$(4.4) \qquad x_o = t, \quad u_o = \partial\phi/\partial t$$

where by (2.7) $u_o = u_o(\mu)$. Assume that the

$$(4.5) \ \text{rank of} \left\| \partial u_I/\partial\mu_\alpha \right\| = n,$$

where I = 0, 1, 2, 3.

By analogy with the usage in [5], the image $u_I$ of $x_I$ under the mapping $x_I \longrightarrow u_I$ will be called its hodograph. Now (4.1) defines an n-dimensional subspace in the hodograph space. The flow will be said to be a single, double, or triple wave accordingly as n=1, 2, or 3; n-dimensional subspaces of either the hodograph space or of space time will be called curves for n = 1; surfaces for n = 2 and hypersurfaces for n = 3. If $n \leqq 3$ the hodograph and

the flow will be said to be degenerate. With no loss of generality assume that $1 \leqq n \leqq 3$. Now let $k(x, t) = \phi - x_I u_I(\mu)$. Then $\partial k/\partial x_J = -(x_I \partial u_I/\partial \mu_\alpha)(\partial \mu_\alpha/\partial x_J)$, and the rank of $\| \partial k/\partial x_J, \partial \mu_\alpha/\partial x_J \| = n$ also, which implies $k = k(\mu)$. Now

(4.6) $\phi = k(\mu) + x_I u_I(\mu)$

and by (4.2)

(4.7) $x_I \partial u_I/\partial \mu_\alpha + \partial k/\partial \mu_\alpha = 0$

whence

(4.8) $(x_I\, \partial^2 u_I/\partial \mu_\alpha \partial \mu_\beta + \partial^2 k/\partial \mu_\alpha \partial \mu_\beta)\partial \mu_\beta/\partial x_J = -\partial\, u_J/\partial \mu_\alpha$

Again by (4.2) the

(4.9) rank of $\| x_I\, \partial^2\, u_I/\partial \mu_\alpha \partial \mu_\beta + \partial^2 k/\partial \mu_\alpha\, \partial \mu_\beta \| = n$

Hereafter, assume that $u_I(\mu)$ and $k(\mu)$ have been chosen to satisfy (4.9). Than (4.7) can be inverted to yield $\mu_\alpha(x)$.

By (4.1) a point on the hodograph is determined by setting $\mu_\alpha = \mu_{\alpha o}$. The set of points in space-time which is mapped onto $u_I(\mu_o)$ will be called its prototype. (4.7) implies

Theorem 4.1: Suppose the coordinate axes of space-time and the hodograph spaces are parallel. Then the prototype of a point $u_I(\mu_o)$ on an n-dimensional hodograph is the $(4 - n)$ - dimensional intersection of n hyperplanes (4.7) with normals parallel to the tangent vectors $\partial\, u_I/\partial \mu_\alpha$ at $u_I\, (\mu_o)$.

Let $D_\alpha = (\partial u_i/\partial \mu_\alpha)(\partial u_i/\partial \mu_\alpha)$, where $\alpha$ is not summed. If $D_\beta = 0$, but $\partial\, u_o/\partial \mu_\beta \neq 0$ for some $\beta$, then (4.7) defines a $(4 - n)$ - dimensional region of $x_1 x_2 x_3$-space at time $t = 1\, (\partial k/\partial \mu_\beta)/(\partial u_o/\partial \mu_\beta)$. If $D_\alpha \neq 0$ for all $\alpha$, then (4.7) corresponds to the intersection of a set of n planes in $x_1 x_2 x_3$-space which move with constant velocities $-(\partial u_o/\partial \mu_\alpha)(\partial u_i/\partial \mu_\alpha)/D_\alpha$, where $\alpha$ is not summed. That is the points $x_i$ that bear given values $u_i(\mu_o)$, $p(\mu_o)$, and $\rho\, (\mu_o)$ move with a constant velocity that depends on $\mu_{\alpha o}$. This suggests

examining the possibility, considered at length in Section 8, of seeking flows with uniformly expanding geometry, in which all flow functions depend only on $x_i/t$, so-called <u>unsteady conical flows</u> [2] or <u>pseudo-stationary flows</u>[1] [4,6] .

So far $\phi$ has merely been compelled to yield a degenerate map. For an unsteady compressible flow (2.7) and (2.8) must also be satisfied. In (2.8) $\partial^2 \phi/\partial x_I \partial x_J$ is required. By (4.3) and (4.4)

$$\partial^2 \phi/\partial x_I \partial x_J = (\partial u_I/\partial \mu_\alpha)(\partial \mu_\alpha/\partial x_J) = (\partial u_J/\partial \mu_\alpha)(\partial \mu_\alpha/\partial x_I) \tag{4.10}$$

Let $\nu_{I\Gamma}(\mu)$ be 4 - n mutually orthogonal unit normals[2] to $u_I = u_I(\mu)$.

That is,

$$\nu_{I\Gamma}\, \partial u_I/\partial \mu_\alpha = 0, \qquad \nu_{I\Gamma}\, \nu_{I\Delta} = \delta_{\Gamma\Delta} \tag{4.11}$$

By (4.10) $(\partial u_I/\partial \mu_\alpha)(\partial \mu_\alpha/\partial x_J)\nu_{J\Gamma} = 0$, and by (4.5)

$$(\partial \mu_\alpha/\partial x_J)\nu_{J\Gamma} = 0 \tag{4.12}$$

By (4.5) and (4.11)

$$\partial \mu_\alpha/\partial x_J = A_{\alpha\beta}\, \partial u_J/\partial \mu_\beta \tag{4.13}$$

for some $A_{\alpha\beta}$. Now by (4.8), (4.10), and (4.5) $(x_I \partial^2 u_I/\partial^2 u_I/\partial \mu_\alpha \partial \mu_\beta + \partial^2 k/\partial \mu_\alpha \partial \mu_\beta) A_{\beta\gamma} = -\delta_{\alpha\gamma}$. The general solution of (4.7) is of the form

$$x_I = x_{I0} + r_\Gamma \nu_{I\Gamma} \tag{4.14}$$

where $x_{I0}$ is a particular solution, and $r_\Gamma$ are independent of the $\mu$'s and each other. Write

$$x_{I0} = A_\alpha\, \partial u_I/\partial \mu_\alpha \tag{4.15}$$

---

2. To summarize the conventions regarding indices in the following discussion, Roman lower case range from 1 to 3; Roman capitals from 0 to 3; Greek lower case from 1 to n; and Greek capitals from 1 to 4 - n.

Then by (4.7)

$$A_\alpha (\partial u_I/\partial \mu_\alpha)(\partial u_I/\partial \mu_\beta) + \partial k/\partial \mu_\beta = 0 \tag{4.16}$$

defines $A_\alpha(\mu)$, and

$$\left[(x_{10} + r_\Gamma \nu_{I\Gamma})\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta + \partial^2 k/\partial\mu_\alpha\partial\mu_\beta\right] A_{\beta\gamma} = -\delta_{\alpha\gamma} \tag{4.17}$$

defines $A_{\beta\gamma}(\mu, r)$. Finally, by (2.8), (4.10), and (4.13)

$$\Big\{a^2\left[(\partial u_I/\partial\mu_\alpha)(\partial u_I/\partial\mu_\beta) - (\partial u_o/\partial\mu_\alpha)(\partial u_o/\partial\mu_\beta)\right] - (\partial u_o/\partial\mu_\alpha + u_i\partial u_i/\partial\mu_\alpha)(\partial u_o/\partial\mu_\beta + u_i\partial u_i/\partial\mu_\beta)\Big\} A_{\alpha\beta} = 0 \tag{4.18}$$

5. Simple waves. When $n = 1$, $A_{\alpha\beta}$ consists of a single element, and (4.18) yields

$$(u_o' + u_i u_i')^2 = a^2 u_i' u_i' \tag{5.1}$$

where primes denote ordinary derivatives with respect to $\mu_1$, and by Bernoulli's equation

$$a^2 = -(\gamma - 1)(u_o + \tfrac{1}{2}u_i u_i) \tag{5.2}$$

If $a^2 \neq 0$ (5.1) and (5.2) imply

$$u_i' u_i' = 4a'^2/(\gamma - 1)^2 \tag{5.3}$$

Now let

$$d\sigma/d\mu_1 = (u_i' u_i')^{\frac{1}{2}} = \pm 2a'/(\gamma - 1) \tag{5.4}$$

to obtain

$$\sigma \pm 2a/(\gamma - 1) = \sigma_o \pm 2a_o/(\gamma - 1) \tag{5.5}$$

where $\sigma_o$ and $a_o$ are constants. Note that $\sigma$ is simply the arc length of the projection of $u_I = u_I(\mu)$ onto any hyperplane $u_o$ = constant. In a one-dimensional simple wave (5.5) reduces to the familiar form (3.25) with $s$ = constant. Since (5.1) is of the form (3.27), the prototype

hyperplanes are characteristic hypersurfaces. Since $u_I$, $\rho$, and s are constant thereon, (3.28) is satisfied. Further note that each prototype hyperplane corresponds to a plane moving with constant velocity

$$-u'_o(\mu)u'_i(\mu)/u'_j(\mu)u'_j(\mu) = \left[u_j u'_j (u'_k u'_k)^{-\frac{1}{2}} \pm a\right] u'_i (u'_m u'_m)^{-\frac{1}{2}}.$$

Since $u_j u'_j (u'_k u'_k)^{-\frac{1}{2}}$ is the component of fluid velocity normal to this plane, its speed is exactly what one would expect in one-dimensional flow.

In general $k(\sigma)$ and two of the functions $u_I(\sigma)$ can be chosen arbitrarily in constructing a simple wave in accordance with (4.7), (5.2), and (5.5).

6. Double waves. When n = 2, let $g_{\alpha\beta}$ be the covariant metric tensor of the surface $u_I = u_I(\mu)$ based on the euclidean metric $ds^2 = du_I du_I$, and let $b^{\epsilon}_{\alpha b}$ be two second fundamental tensors [3]. By definition

$$g_{\alpha\beta} = (\partial u_I/\partial\mu_\alpha)(\partial u_I/\partial \mu_\beta) \tag{6.1}$$

$$b^{\epsilon}_{\alpha\beta} = \nu_{M\epsilon}\, \partial^2 u_M/\partial \mu_\alpha \partial \mu_\beta \tag{6.2}$$

where $\nu_{M\epsilon}$ are two mutually orthogonal unit normals which satisfy (4.11).

Also

$$u_k \partial u_k/\partial \mu_\alpha = q\partial q/\partial \mu_\alpha \tag{6.3}$$

where $q^2 = u_k u_k$. Write $\partial q/\partial\mu_\alpha = q_{,\alpha}$ where the subscript $_{,\alpha}$ denotes the covariant derivatives with respect to $\mu_\alpha$ and based on $g_{\alpha\beta}$

By (4.15), (4.16), and (6.1)

$$x_{10} = -(\partial k/\partial\mu_\gamma)g^{\gamma\delta}(\partial u_I/\partial\mu_\delta) \tag{6.4}$$

where $g^{\gamma\delta}$ is the inverse of $g_{\alpha\beta}$. Since the Christoffel symbols of the first kind, based on $g_{\alpha\beta}$, are $[\alpha\ \beta, \gamma] = (\partial u_I/\partial\mu_\gamma)(\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta)$, the second covariant derivative of k becomes

$$k_{,\alpha\beta} = \partial^2 k/\partial\mu_\alpha\partial\mu_\beta - (\partial k/\partial\mu_\gamma)g^{\gamma\delta}(\partial u_I/\partial\mu_\delta)(\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta) \tag{6.5}$$

Now (4.17) becomes $(k_{,\alpha\beta} + r_\epsilon b^{\epsilon}_{\alpha\beta})A_{\beta\gamma} = -\delta_{\alpha\gamma}$. Since (4.18) is homogeneous

in $A_{\alpha\beta}$, only the adjoint of $(k_{,\alpha\beta} + r_\epsilon b^\epsilon_{\alpha\beta} = (-1)^{\alpha+\beta}(k_{,\alpha+1\ \beta+1} + r_\epsilon b^\epsilon_{\alpha+1\ \beta+1})$ is required. Then since all four of $r_\epsilon$ and $\mu_\alpha$ are independent, (4.18) yields

$$(6.6)\quad \left[a^2(g_{\alpha\beta}-u_{o,\alpha}u_{o,\beta}) - (u_{o,\alpha}+qq_{,\alpha})(u_{o,\beta}+qq_{,\beta})\right](-1)^{\alpha+\beta}\, b^\epsilon_{\alpha+1\ \beta+1} = 0$$

$$(6.7)\quad \left[a^2(g_{\alpha\beta}-u_{o,\alpha}u_{o,\beta}) - (u_{o,\alpha}+qq_{,\alpha})(u_{o,\beta}+qq_{,\beta})\right](-1)^{\alpha+\beta} k_{;\alpha+1\ \beta+1} = 0$$

Note that the

$$(6.8)\quad \text{rank of } \left\| a^2(g_{\alpha\beta} - u_{o,\alpha}u_{o,\beta}) - (u_{o,\alpha} + qq_{,\alpha})(u_{o,\beta} + qq_{,\beta}) \right\| > 0$$

For, suppose the rank were zero. Then $u_{o,\alpha} + qq_{,\alpha} \neq 0$ for some $\alpha$ since $g_{\alpha\beta}$ is non-singular. Let $L_\alpha$ be a non-trivial solution of $(u_{o,\alpha} + u_i u_{i,\alpha})L_\alpha = 0$. Now rank zero implies $(u_{i,\alpha} L_\alpha)(u_{i,\beta}L_\beta) = 0$, whence $u_{i,\alpha}L_\alpha = 0$. Thus $u_{i,\alpha} = A_i(u_{o,\alpha} + u_j u_{j,\alpha})$ for some $A_i$. If $u_i A_i = 0$, then $u_{i,\alpha} = A_i u_{o,\alpha}$, and $g_{\alpha\beta} = (1 + A_i A_i)\, u_{o,\alpha}u_{o,\beta}$, which is singular. If $u_i A_i = 1$, then $u_{o,\alpha} = 0$, so $u_o = \partial\phi/\partial t$ = constant, and $\phi = u_o t + f(x_1, x_2, x_3)$ which yields a steady flow field, to be excluded from the present discussion. If $u_i A_i \neq 0$ or $1$, $u_{i,\alpha} = A_i\, u_{o,\alpha}/(1 - u_j A_j)$, whence $(1 - u_i A_i)^2 g_{\alpha\beta} = \left[(1 - u_i A_i)^2 + A_i A_i\right] u_{o,\alpha}u_{o,\beta}$ is singular.

(6.6) is a system of two second order quasilinear partial differential equations for four functions. To determine $u_I(\mu)$ requires two more equations which may be obtained by assigning a special form to the coefficient tensor of (6.6). The resulting systems are classified according to the nature of the characteristic curves of their integral surfaces. A characteristic is a curve on which the coordinate functions, their first partial derivatives, and hence the metric tensor are continuous, while the components of the second fundamental tensors may have discontinuities. Suppose $\partial u_i/\partial\mu_\alpha$ are known along $\mu_\alpha = \mu_\alpha(t)$ on $u_I = u_I(\mu)$. By (6.2) the strip conditions $d(\partial u_M/\partial\mu_\alpha)/dt = (\partial^2 u_M/\partial\mu_\alpha\partial\mu_\beta)d\mu_\beta/dt$ imply

$$(6.9)\quad b^\epsilon_{\alpha\beta}\, d\mu_\beta/dt = \nu_{M\epsilon}\, d(\partial u_M/\partial\mu_\alpha)/dt$$

Then $b^\epsilon_{\alpha\beta}$ fail to be uniquely determined along $\mu_\alpha = \mu_\alpha(t)$ by (6.6) and (6.9) if and only if

$$(6.10)\quad \left[a^2(g_{\alpha\beta} - u_{o,\alpha}u_{o,\beta}) - (u_{o,\alpha}+ qq_{,\alpha})(u_{o,\beta}+ qq_{,\beta})\right](d\mu_\alpha/dt)(d\mu_\beta/dt) = 0$$

This defines the characteristic directions $d\mu_\alpha/dt$. By (6.5), if $u_I(\mu)$ are known, then (6.7) is a linear partial differential equation for k which also has the characteristic directions (6.10). Equations (6.6) and (6.7) will be said to be of hyperbolic, parabolic, or elliptic type whenever $\Omega =$ det $\left\| a^2(g_{\alpha\beta} - u_{o,\alpha}\, u_{o,\beta}) - (u_{o,\alpha} + qq_{,\alpha})(u_{o,\beta} + qq_{,\beta}) \right\| < 0, = 0, > 0$. Hereafter (6.6) and (6.7) will be assumed to be hyperbolic.

If $\sigma_c$ is the arc-length of a characteristic, then (6.10) becomes

$$(6.11) \qquad (du_o/d\sigma_c + qdq/d\sigma_c)^2 = a^2 \left[1 - du_o/d\sigma_c)^2\right]$$

which is identical with (5.1). Hence

Theorem 6.1: The characteristics of the hodographs of double waves are one-dimensional hodographs.

By (6.11) the component of $u_{o,\alpha} + qq_{,\alpha}$ along either characteristic is $(u_{o,\alpha} + qq_{,\alpha})\, d\mu_\alpha/d\sigma_c = \pm\, a \left[1 - (du_o/d\sigma_c)^2\right]^{\frac{1}{2}}$. Hence

Theorem 6.2: On the hodographs of double waves the curves of constant speed of sound and their orthogonal trajectories bisect the angles between the characteristics.

In general, curves $x_I = x_I(t)$ (where t is any parameter, not necessarily time) in space-time are mapped onto curves $\mu_\alpha = \mu_\alpha(t)$ on the hodograph. It is convenient to know the relation between tangent vectors of a pair of corresponding curves. By (4.7), (4.14), (6.4), and (6.5) (6.12)

$$(6.12) \qquad (dx_I/dt)(\partial u_I/\partial\mu_\alpha) + \left[r_\epsilon(t) b^\epsilon_{\alpha\beta} + k_{,\alpha\beta}\right] d\mu_\beta/dt = 0$$

for some $r_\epsilon(t)$. Unless det $\left\| r_\epsilon\, b^\epsilon_{\alpha\beta} + k_{,\alpha\beta} \right\| = 0$, this determines $d\mu_\beta/dt$.

Conversely, if the curve $\mu_\alpha = \mu_\alpha(t)$ is given on a hodograph surface, its prototype is the hypersurface

$$(6.13) \qquad x_I(r, t) = x_{I0}(t) + \left[r_\epsilon - A_\epsilon(t)\right] \nu_{I\epsilon}(t)$$

where the normals $\nu_{I\epsilon}$ to the hodograph satisfy (4.11), $x_{I0}(t)$ is defined by (6.4), and $A_\epsilon(t)$ by $dA_\epsilon/dt = \nu_{I\epsilon}\, dx_{I0}/dt$ (to make the three families of curves of parameters $r_1$, $r_2$ or t mutually orthogonal, as a matter of convenience). For (6.13) an analog of (6.12) is

$$(6.14) \qquad (\partial x_I/\partial t)(\partial u_I/\partial\mu_\alpha) + \left[(r_\epsilon - A_\epsilon) b^\epsilon_{\alpha\beta} + k_{,\alpha\beta}\right] d\mu_\beta/dt = 0$$

Since $\nu_{I\epsilon}\partial x_I/\partial t = 0$, (6.14) implies

(6.15) $$\partial x_I/\partial t = -(\partial u_I/\partial\mu_\alpha)g^{\alpha\beta} \quad (r_\epsilon - A_\epsilon)b^\epsilon_{\alpha\beta} + k_{,\beta\gamma} \quad d\mu_\gamma/dt$$

In general the direction of $\partial x_I/\partial t$ varies with $r_1$ and $r_2$ on the prototype of $\mu_\alpha(t)$, so the hypersurface (6.13) need not be flat. This raises the question, what curves on the hodograph have flat prototypes? Since $\partial^2 x_I/\partial r_\alpha \partial r_\beta = 0$, (6.13) will be flat if and only if [3]

(6.16) $$\det \left\| \partial x_I/r_1,\ \partial x_I/\partial r_2,\ \partial x_I/\partial t,\ \ \partial^2 x_I/\partial r_\epsilon \partial t \right\| = 0$$

The product of (6.16) and $\det \left\| \nu_{I\alpha},\ \partial u_i/\partial\mu_\beta \right\|$ yields

(6.17) $$\text{rank of } \left\| b^i_{\beta\gamma}\ d\mu_\gamma/dt \right\| \leqq 1$$

where $b^3_{\alpha\beta} \equiv k_{,\alpha\beta}$. Now let $n_\beta\ d\mu_\beta/dt = 0$, $n_\beta n_\beta = 1$. Then for some $B_i$, $C_i$, and $D_i$, $b^i_{\beta\gamma} = B_i n_\beta n_\gamma + C_i(n_\beta d\mu_\gamma/dt + n_\gamma d\mu_\beta/dt + D_i(d\mu_\beta/dt)(d\mu_\gamma/dt)$. By (6.17) the rank of $\left\| C_i\ n_\beta + D_i\ d\mu_\beta/dt \right\| \leqq 1$. Hence $C_i = CE_i$, $D_i = DE_i$ for some $C$, $D$, and non-null $E_i$. Now

(6.18) $$b^i_{\beta\gamma} = B_i n_\beta n_\gamma + E_i \left[ C(n_\beta d\mu_\gamma/dt + n_\gamma\ d\mu_\beta/dt) + D(d\mu_\beta/dt)\ (d\mu_\gamma/dt) \right]$$

Suppose $B_i$ and $E_i$ are linearly independent. Let $F_{i\alpha}$ be two linearly independent solutions of $E_i F_{i\alpha} = 0$. Then $F_{i\epsilon} b^i_{\alpha\beta} = (B_i F_{i\epsilon}) n_\alpha n_\beta$, (6.6), and (6.7) imply (6.10), i.e. $\mu_\alpha = \mu_\alpha(t)$ is a characteristic.

Next show that the prototypes of both families of characteristics are flat. Let $\mu_\alpha = \mu^\epsilon_\alpha(t)$ define one characteristic from each family through a given point P of the hodograph. At P, by (6.18)

$$2\left[a^2(g_{\alpha\beta} - u_{o,\alpha}u_{o,\beta}) - (u_{o,\alpha} + qq_{,\alpha})(u_{o,\beta} + qq_{,\beta})\right] =$$
$$(-1)^{\alpha+1}\left[(d\mu_{\alpha+1}/dt)(d\mu^2{}_{\beta+1}/dt) + (d\mu^1{}_{\beta+1}/dt)(d\mu^2_{\alpha+1}/dt)\right] f$$

for some $f \neq 0$. Then by (6.6) and (6.7)

(6.19) $$b_{\alpha\beta}\ (d\mu_\alpha{}^{+1}/dt)\ (d\mu^\epsilon_\beta/dt) = 0$$

where $\epsilon$ is not summed. Since these have non-trivial solutions $d\mu^{\epsilon+1}_\alpha/dt$, (6.19) implies (6.17) for $\mu_\alpha = \mu^\epsilon_\alpha$.

Now investigate the relation between tangents to characteristics and unit normals $n_I^\epsilon$ to prototypes $x_I^\epsilon = (r, t)$ of characteristics $\mu_\alpha = \mu_\alpha^\epsilon(t)$. By (6.13) $n_I^\epsilon \nu_{I\gamma} = n_I^\epsilon \partial x_I^\epsilon / \partial r_\gamma = 0$. For some $A_\alpha^\epsilon$

(6.20) $$n_I^\epsilon = A_\alpha^\epsilon \, \partial u_I / \partial \mu_\alpha$$

Since $n_I^\epsilon \partial x_I^\epsilon / \partial t = 0$, then by (6.18)

(6.21) $$A_\alpha^\epsilon \left[ (r_\gamma - A_\gamma) b_{\alpha\beta}^\gamma + k_{,\alpha\beta} \right] d\mu_\beta^\epsilon / dt = 0$$

If

(6.22) $$\det \left\| c_\gamma \, b_{\alpha\beta}^\gamma + k_{,\alpha\beta} \right\| \neq 0$$

where $c_\gamma = r_\gamma - A_\gamma$, then by (6.19) the general solution of (6.21) is a $A_\alpha^\epsilon = g d\mu_\alpha^{\epsilon+1}/dt$ for some g. By (6.20)

(6.23) $$n_I^\epsilon = g(\partial u_I / \partial \mu_\alpha)(d\mu_\alpha)(d\mu_\alpha^{\epsilon+1}/dt)$$

i.e. the tangent to $\mu_\alpha = \mu_\alpha^{\epsilon+1}(t)$ is normal to the prototype of $\mu_\alpha = \mu_\alpha^\epsilon$.

These considerations and elementary calculation yield

Theorem 6.3: _If for a double wave the rank of_ $\left\| b_{\alpha\beta}^1, \; b_{\alpha\beta}^2, \; k_{,\alpha\beta} \right\| = 2$, _then_

(1) _The characteristics are the only curves on the hodograph with flat prototypes._

(2) _The tangent at any point P of a characteristic is parallel to the normal, at points on the prototype of P, to the prototype hypersurface of the other characteristic through P._

(3) _The prototype of a characteristic is a (sound wave) characteristic hypersurface._

For the omitted cases, first suppose (6.22) is false for all $c_\gamma$. Then $b_{\alpha\beta}^\epsilon = K_\epsilon B_\alpha B_\beta$ and $k_{,\alpha\beta} = K B_\alpha B_\beta$ for some $B_\alpha$, $K_\epsilon$, and K. Now the Riemann tensor of $u_I = u_I(\mu)$ is

(6.24) $$R_{\alpha\beta\gamma\delta} = b_{\alpha\gamma}^\epsilon b_{\beta\delta}^\epsilon - b_{\alpha\delta}^\epsilon b_{\beta\gamma}^\epsilon$$

[3], where $\epsilon$ is summed. But now $R_{\alpha\beta\gamma\delta} = 0$, so the parameters $\mu\alpha$ can be chosen so that

(6.25) $$g\alpha\beta = \delta_{\alpha\beta}$$

which implies

(6.26) $(\partial u_I/\partial_{\mu\gamma})(\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta) = 0$

so by (4.11), (6.2), (6.25), and (6.26) $\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta = \nu_I\, K\, B_\alpha\, B_\beta$ and by (6.5) $\partial^2 k/\partial\mu_\alpha\partial\mu_\beta = KB_\alpha B_\beta$. Hence the rank of $\| x_I\partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta + \partial^2 k/\partial\mu_\alpha\partial\mu_\beta \| \leqq 1$, contradicting (4.9).

Next suppose the rank of $\| b^1_{\alpha\beta}, \quad b^2_{\alpha\beta}, \quad k_{,\alpha\beta} \| < 2$. Then

(6.27) $$b^i_{\alpha\beta} = E_i\, a_{\alpha\beta}$$

for some $E_i$ and $a_{\alpha\beta}$. If $E_\alpha E_\alpha = 0$, then

(6.28) $$b^\epsilon_{\alpha\beta} = \nu_{I\epsilon}\, \partial^2 u_I/\partial\mu_\alpha\partial\mu_\beta = 0$$

which implies $R_{\alpha\beta\gamma\delta} = 0$. Choose parameters such that (6.25) and (6.26) hold. Then $\partial^2 u_I/\partial\mu_\alpha\,\partial\mu_\beta = 0$, and

(6.29) $$u_I = a_{I\alpha}\mu_\alpha + b_I$$

where $a_{I\alpha}$ and $b_I$ are constants with $a_{I\alpha}\, a_{I\beta} = \delta_{\sigma\beta}$. Since $\nu_{I\epsilon}$ may be chosen to have constant components, the hodograph is the intersection of the two hyperplanes $\nu_{I\epsilon}(u_I - b_I) = 0$. This implies that the velocity potential must be of the form

(6.30) $$\phi = b_I x_I + F(\psi_1, \psi_2)$$

where $\psi_\alpha = a_{I\alpha}\, x_I$ and $F$ is arbitrary. If both $a_{o\alpha} = 0$, the flow is a steady plane flow in planes normal to the lines $a_{i\alpha}(x_i - x_i^*) = 0$. Now assume $a_{o\alpha}$ is non-null. Since the rank of $a_{I\alpha}$ is two, rotate space axes to obtain the same form for $\phi$ with $\psi_1 = t \cos\alpha + x_i \sin\alpha$ $\psi_2 = x_2 \sin\beta + (x_1 \cos\alpha - t\sin\alpha)\cos\beta$ for constants $\alpha$ and $\beta$, and suitably redefined $b_I$. Since $\partial\phi/\partial x_3 = u_3 = b_3$, these are swept $x_1x_2$ - plane flows. If $\sin\alpha\, \sin\beta \neq 0$ the basic plane flow is any plane flow steady relative to axes translated with uniform velocity $x_1'$, $x_2'$, 0 defined by $\cos\alpha + x_1' \sin\alpha = 0$, $(x_1' \cos\alpha - \sin\alpha)\cos\beta + x_2' \sin\beta = 0$. Now $\mu_\alpha = \partial F/\partial\psi_\alpha$.

If $\sin\alpha \sin\beta = 0$, by a rotation of $x_1 x_2$ - axes at most, $\phi$ can be reduced to the form $\phi = b_I x_I + F(x_1, t)$, which corresponds to any swept unsteady one-dimensional flow.

Finally, suppose $E_\alpha E_\alpha \neq 0$. Then $c_\epsilon b^\epsilon_{\alpha\beta} = 0$ for non-trivial solutions of $c_\epsilon E_\epsilon = 0$. Since $u_o$ = constant leads to steady flows, assume $\mu_1 = u_o$, $\mu_2 = u_3$. Then $c_\epsilon \nu_{\gamma\epsilon} \partial^2 u_\gamma/\partial\mu_\alpha\partial\mu_\beta = 0$. If both $c_\epsilon \nu_{\gamma\epsilon} = 0$, then $\nu_{\gamma\epsilon} = r_\gamma E_\epsilon$ for some $r_\gamma$. Now $\nu_{o\epsilon} = - E_\epsilon r_\alpha \partial u_\alpha/\partial\mu_1$ and $\nu_{3\epsilon} = - E_\epsilon r_\alpha \partial u_\alpha/\partial\mu_2$. Then $\delta_{\delta\epsilon} = \nu_{I\delta}\nu_{I\epsilon} = E_\delta E_\epsilon \left[(r_\alpha \partial u_\alpha \partial u_\alpha/\partial\mu_1)^2 + r_\alpha r_\alpha + (r_\alpha \partial u_\alpha/\partial\mu_2)^2\right]$ is impossible. Hence assume

$$\partial^2 u_2/\partial\mu_\alpha\partial\mu_\beta = c\partial^2 u_1/\partial\mu_\alpha\partial\mu_\beta \tag{6.31}$$

If c is a constant, then by (6.31) the hodograph lies entirely in a hyperplane $a_I(u_I - b_I) = 0$ for some constants $a_I$ and $b_I$, with $a_i a_i \neq 0$. After a rotation of space axes this yields $\partial\phi/\partial x_3 - A\partial\phi/\partial t = B$ for some constants A and B, with the general solution $\phi = Bx_3 + F(x_1, x_2, Ax_3 + t)$. If $A = 0$ this is a swept unsteady plane flow, which still has to be chosen to have a two-dimensional hodograph. If $A \neq 0$, the flow will be steady relative to axes translated with uniform velocity $0, 0, 1/A$. If c is not constant, the integrability conditions of (6.31), $(\partial c/\partial\mu_\gamma)(\partial^2 u_1/\partial\mu_\alpha\partial\mu_\beta) = (\partial c/\partial\mu_\beta)(\partial^2 u_1/\partial\mu_\alpha\partial\mu_\gamma)$ yield $\partial^2 u_\gamma/\partial\mu_\alpha\partial\mu_\beta = a_\gamma B_\alpha B_\beta$ for some $B_\alpha = \partial c/\partial\mu_\alpha$ and some $a_\gamma$, whence $b^\epsilon_{\alpha\beta} = K_\epsilon B_\alpha B_\beta$ for some $K_\epsilon$. If $K_\epsilon K_\epsilon B_\alpha B_\alpha \neq 0$, then by (6.27) $k_{,\alpha\beta} = KB_\alpha B_\beta$ for some K, which yields a previously obtained contradicition. If $K_\epsilon K_\epsilon B_\alpha B_\alpha = 0$, then $b^\epsilon_{\alpha\beta} = 0$, a case discussed in the preceding paragraph.

The foregoing discussion and an application of (6.17) yield

Theorem 6.4: _If for a double wave the rank of_ $\left\| b^1_{\alpha\beta}, \ b^2_{\alpha\beta}, \ K_{,\alpha\beta} \right\| \leqq 1$, _then_

(1) _The flow is a uniformly translated swept steady plane flow or a swept unsteady one-dimensional flow._

(2) _Any curve on the hodograph has a flat prototype._

(3) _Conclusions (2) and (3) of Theorem 6.3 apply to these flows._

7. <u>Triple waves</u>. When $n = 3$, let $g_{ij}$ be the covariant metric tensor of the hypersurface $u_I = u_I(\mu)$ based on the euclidean metric $ds^2 = du_I du_I$, and let $b_{ij}$ be its second fundamental tensor [3] By definition

$$(7.1) \qquad g_{ij} = (\partial u_I/\partial\mu_i)(\partial u_I/\partial\mu_j)$$

$$(7.2) \qquad b_{ij} = \nu_I \partial^2 u_I/\partial\mu_j\,\partial\mu_i$$

where $\nu_I = \nu_{II}$ is the unit normal defined to within sign by (4.11). Also

$$(7.3) \qquad u_k \partial u_k/\partial\mu_i = q\partial q/\partial\mu_i$$

where $q^2 = u_k u_k$. Write $\partial q/\partial\mu_i = q_{,i}$, where the subscript $_{,i}$ denotes the covariant derivative with respect to $\mu_i$ and based on $g_{ij}$.

By (4.15), (4.16), and (7.1)

$$(7.4) \qquad x_{10} = -(\partial k/\partial\mu_i)g^{ij}(\partial u_I/\partial\mu_j)$$

where $g^{hi}$ is the inverse of $g_{jk}$. Since the Christoffel symbols of the first kind, based on $g_{ij}$, are $[ij, k] = (\partial u_I/\partial\mu_k)(\partial^2 u_I/\partial\mu_i\partial\mu_j)$, the second covariant derivative of k becomes

$$(7.5) \qquad k_{,ij} = \partial^2 k/\partial\mu_i\partial\mu_j - (\partial k/\partial\mu_h)g^{hk}(\partial u_I/\partial\mu_k)(\partial^2 u_I/\partial\mu_i\partial\mu_j)$$

Now (4.17) becomes $(k_{,ij} + rb_{ij})A_{jk} = -\delta_{ik}$, where $r = r_1$. Since (4.18) is homogeneous in $A_{ij}$, only the

$$(7.6) \qquad \text{adjoint of } (k_{,ij} + rb_{ij}) = r^{m-1}R^m_{ij}$$

is required, where

$$2R^1_{ij} = R(k_{,ij}, k_{,ij}), \quad R^2_{ij} = R(k_{,ij}, b_{ij}), \quad 2R^3_i{}_j = R(b_{ij}, b_{ij})$$

$$(7.7) \qquad R(k_{,ij}, b_{ij}) = k_{,i+1j+1}b_{i+2\,j+2} - k_{,i+1j+2}b_{i+2j+1}$$
$$+ k_{,i+2j+2}b_{i+1+1j+1} - k_{,i+2j+1}b_{i+1j+2}$$

Since all four of r and $\mu_i$ are independent, (4.18), (7.6), and (7.7) yield

$$(7.8) \qquad \left[a^2(g_{ij} - u_{o\,,i}u_{o,j}) - (u_{o,i} + qq_{,i})(u_{o,j} + qq_{,j})\right]R^m_{ij} = 0$$

Since $R^m_{ij}$ is a relative contravariant tensor of weight , the system (7.8) may be considered to be tensor equations. Furthermore, note

that

(7.9) $R^3_{ij} = b_{i+1j+1}b_{i+2j+2} - b_{i+1j+2}b_{i+2j+1} = R_{i+1i+2j+1j+2}$

are the components of the Riemann tensor of the hodograph hypersurface.

(7.8) is a system of three second order partial differential equations for the five functions $u_I$ and $k$. If $k \neq (=) 0$, to determine $u_i(\mu)$ and $k(\mu)$ requires two (three) additional equations, which may be obtained by assigning a special form to the coefficient tensor of (7.8). The resulting systems are classified according to the nature of the characteristic surfaces of their internal hypersurfaces. In the present context a characteristic is a surface on which the coordinate functions, their first partial derivatives, and hence the metric tensor, are continuous, while the components of the second fundamental tensor may have discontinuities there.

One might hope, by analogy with Theorem 6.1, that the characteristics of the hodographs of triple waves would be two-dimensional hodographs. Although it has not been possible to prove this, it will be shown that two of the equations for these characteristics are consequences of the equations (6.6) and (6.7) for two-dimensional hodographs.

To find the characteristic conditions corresponding to (7.8), consider a triple of functions $M_i(t_1, t_2)$ of class $C_2$ with Jacobian matrix $\| \partial M_i/\partial t_\alpha \|$ of rank two. Assume that on the surface

(7.10) $\mu_i = M_i(t_1, t_2)$

the values of

(7.11) $u_i\,(M(t)) = U(t)$, $k(M(t)) = K(t)$

(7.12) $\partial u_I/u_i = P_{Ii}(t)$, $\partial k/\partial\mu_i = Q_i(t)$

are known. Also on this surface let $g_{ij} = G_{ij}$, $b_{ij} = B_{ij}{}^{(t)}$, and $k,_{ij} = K,_{ij}{}^{(t)}$ be the values of the three dimensional metric tensor, second fundamental tensor, and covariant derivative of k, which will now be calculated. Define non-trivial $L_i$, $N_I$, and $N_{I\gamma}$ by

(7.13) $N_I\,P_{Ii} = 0$, $N_I\,N_I = 1$

(7.14) $N_{I\gamma}\,\partial u_I/\partial t_\alpha = (N_{I\gamma}\,P_{Ii})\,\partial M_i/\partial t_\alpha = 0, \alpha, \gamma = 1, 2.$

$N_{I1}N_{I2} = 0$

(7.15) $L_j\,\partial M_j/\partial t_\alpha = 0$

By (7.13) and (7.14)

(7.16) $N_I = C_\epsilon N_{I\epsilon}$ $\qquad \epsilon = 1, 2$

for some $C_\epsilon$ (t). Since

(7.17) $\partial^2 {}_I/\partial t_\alpha \partial t_\beta = (\partial^2 u_I/\partial\mu_i\partial\mu_j)(\partial M_i/\partial t_\alpha)(\partial M_j/\partial t_\beta) + P_{Ii}\, \partial^2 M_i/\partial t_\alpha\, \partial t_\beta$

then by (7.2)

(7.18) $B_{ij}(\partial M_i/\partial t_\alpha)\, (\partial M_j/\partial t_\beta) = N_I \partial^2\, U_I/\partial t_\alpha\, \partial t_\beta$

Also (7.13) implies

(7.19) $B_{ij}\, \partial M_j/\partial t_\alpha = - P_{Ii}\, \partial N_I/\partial t_\alpha$

Since $\partial M_i/\partial t_\alpha$ and $L_i$ are independent

(7.20) $B_{ij} = H_{\alpha\beta}\, (\partial M_i/\partial t_\alpha)\, (\partial M_j/t_\beta) + H_\alpha\, (L_j \partial M_i/\partial t_\alpha + L_i\, \partial M_j/\partial t_\alpha) + BL_i L_j$

for some $H_{\alpha\beta}$, $H_\alpha$, and B. Let

(7.21) $h_{\alpha\beta} = (\partial M_i/\partial t_\alpha)\, (\partial M_i/\partial t_\beta)$, $h_{\alpha\beta} h^{\beta\gamma} = \delta^\gamma{}_\alpha$

(7.22) $b^{\epsilon *}_{\alpha\beta} = N_{I\epsilon}\, \partial^2\, U_I/\partial t_\alpha\, \partial t_\beta$

Then by (7.20), (7.18), and (7.19)

(7.23) $H_{\alpha\beta} = C_\epsilon\, b^{\epsilon *}_{\alpha\beta}\, h^{\alpha\gamma}\, h^{\beta\delta}$

(7.24) $H_\alpha\, L_i h_{\alpha\gamma} + H_{\alpha\beta}\, h_{\beta\gamma}\, \partial M_i/\partial t_\alpha + P_{Ii}\, \partial N_I/\partial t_\gamma = 0$

Since these determine $H_{\alpha\beta}$, and $H_\alpha$, everything in the right hand member of (7.20) except B can be found in terms of surface data on (7.10).

Next, consider $K_{,ij}$. First note that

(7.25) $\partial P_{Ii}/\partial t_\alpha = (\partial^2 u_I/\partial\mu_i\partial\mu_j)\, (\partial M_j/\partial t_\alpha)$

By (7.5) and (7.25)

(7.26) $K_{,ij}\, \partial M_j/\partial t_\alpha = \partial Q_i/\partial t_\alpha - Q_h G^{hk} P_{Ik} \partial P_{Ii}/\partial t_\alpha$

Now let $K^*;_{\alpha\beta}$ be the second covariant derivative of $K(t)$ with respect to $t_\alpha$, $t_\beta$ based on $g^*_{\alpha\beta} = (\partial U_I/\partial t_\alpha)(\partial U_I/\partial t_\beta)$.

By (6.5)

$$(7.27)\quad K^*;_{\alpha\beta} = \partial^2 K/\partial t_\alpha \partial t_\beta - (\partial K/\partial t_\gamma) g^{*\gamma\delta} (\partial U_I/\partial t_\delta) \partial^2 U_I/\partial t_\alpha \partial t_\beta$$

By (7.5), (7.25), and (7.27)

$$(7.28)\quad K,_{ij} (\partial M_i/\partial t_\alpha)(\partial M_j/\partial t_\beta) - K^*;_{\alpha\beta} =$$

$$- (Q_k G^{kh} P_{ih} - \partial K/\partial t_\gamma\, g^{*\gamma\delta} \partial U_I/\partial t_\delta)\, \partial^2 U_I/\partial t_\alpha \partial t_\beta$$

Since the vector in parentheses in the right hand member is normal to $\partial M_i/\partial t_\epsilon$, then for some $S_\epsilon(t)$ $Q_k G^{kh} P_{ih} - \partial K/\partial t_\gamma g^{*\gamma\delta} \partial U_I/\delta t_\delta = S_\epsilon N_{I\epsilon}$

$$(7.29)\quad K,_{ij} (\partial M_i/\partial t_\alpha)(\partial M_j/\partial t_\beta) = K^*;_{\alpha\beta} - S_\epsilon b^\epsilon_{\alpha\beta}$$

As in the case of $B_{ij}$, expand

$$(7.30)\quad K,_{ij} = M_{\alpha\beta}(\partial M/\partial t_\alpha)(\partial M/\partial t_\beta) + M_\alpha(L_j \partial M_i/\partial t_\alpha + L_i \partial M_j/\partial t_\alpha + CL_i L_j$$

for some $M_{\alpha\beta}$, $M_\alpha$, and $C$. Then from

$$M_{\alpha\beta} h_{\alpha\gamma} h_{\beta\delta} = K,_{ij}\, \partial M_i/\partial t_\gamma \partial M_j/\partial t_\delta$$

$$L_i M_\alpha h_{\alpha\gamma} + M_{\alpha\beta} h_{\beta\gamma} = K,_{ij}\, \partial M_j/\partial t_\gamma$$

and with the aid of (7.25) to (7.29) one can find $M_{\alpha\beta}$ and $M_\alpha$ in terms of surface data.

Only B and C remain to be found, if possible, from (4.18) which may be rewritten as

$$F_{ij} \text{ adjoint } (K,_{ij} + r B_{ij}) = 0$$

for all r, where $B_{ij}$ and $K,_{ij}$ are to be evaluated by means of (7.20) and (7.30), where

$$(7.31)\quad F_{ij} = A^2(G_{ij} - P_{o,i} P_{o,j}) - (P_{o,i} + U_h P_{hi})(P_{o,j} + U_k P_{kj})$$

The fundamental equation may also be rewritten in the form

(7.32) $F^1_{ij}$ adjoint $(K^1,_{ij} + B^1_{ij}) = 0$

where $F^1_{hk} = F_{ij}\, J_{hi}\, J_{kj}$, $K^1,_{ij} + r\, B^1_{ij} = (K,_{ij} + rB_{ij})\, J_{hi} J_{kj}$,

and $\| J_{hi} \| = \| \partial M_i/\partial t_1,\ \partial M_i/\partial t_2,\ L_i \|$. Now B and C appear in the adjoint in (7.32) only in the combination $(C + r\,B)\,(L_i L_i)^2$. Then B and C will fail to be uniquely determined if the coefficient of this product vanishes, i.e. if

$$F'_{\alpha\,\beta}(-1)^{\alpha + \beta}\left[K^*;_{\alpha + 1\,\beta + 1} + (r\,C_\epsilon - S_\epsilon)b^{\epsilon *}_{\alpha + 1\,\beta + 1}\right] = 0$$

where $$F_{ij}\ \partial M_i/\partial t_\alpha \partial M_j/\partial t_\beta =$$

$$F'_{\alpha\,\beta} = a^{*2}(g^*_{\alpha\,\beta} - u^*_o;_\alpha\, u^*_o;_\beta$$
$$-(u^*_{o;\alpha} + q^*\, q^*;_\alpha)\,(u^*_{o;\,\beta} + q^* q^*;_\beta)$$

Since r is arbitrary, this yields

(7.33) $C_\epsilon F'_{\alpha\,\beta}\ (-1)^{\alpha + \beta}\ b^{\epsilon *}_{\alpha+1\ \beta + 1} = 0$

(7.34) $F'_{\alpha\,\beta}\ (-1)^{\alpha + \beta}(K^*;_{\alpha + 1\,\beta + 1} - S_\epsilon\ b^\epsilon_{\alpha + 1}) = 0$

Owing to the presence of $C_\epsilon\ S_\epsilon$ (7.33) and (7.34) are implied by but not equivalent to (6.6) and (6.7). The three consistency conditions obtained from (7.33) by requiring the coefficients of 1, r, $r^2$ to vanish and formally setting B = C = 0 may conceivably contain additional information which can be combined with (7.33) and (7.34) to obtain (6.6) and (6.7), but the author has been unable to reach a conclusion on this matter.

It should be remarked that the form of (7.33) can be simplified considerably. By (7.2), (7.13), (7.16), and (7.22) $C_\epsilon b^{\epsilon *}_{\alpha\beta} = N_I \partial^2 u_I/\partial t_\alpha \partial t_\beta = B_{ij} M_{i\alpha} M_{j\beta}$, where $M_{i\alpha} = \partial M_i/\partial t_\alpha$. Then (7.33) becomes

$$F_{rs} B_{ij}(-1)^{\alpha+\beta}\, M_{r\alpha} M_{s\beta} M_{i\alpha+1} M_{j\beta+1} = 0$$

Since the terms with r = i and s = j vanish, this may be rewritten as

(7.35) $(B_{ij}F_{i+1j+1} - B_{i+1j}F_{ij+1} - B_{ij+1}F_{i+1j} + B_{i+1j+1}F_{ij})L_{i+2}L_{j+2} = 0$

where solutions $L_i$ of (7.15) may be defined as

$$(7.36) \quad L_{i+2} = M_{i1}M_{i+12} - M_{12}M_{i+11}$$

In a given triple wave $B_{ij}(M)$ and $F_{ij}(M)$ are known functions of $M_i$. If the form (7.35) is not definite, then as one would expect, (7.35) with $L_i$ defined by (7.36) is a first order partial differential equation for the $M_i(t_1, t_2)$. If $k = 0$, as in the next section, (7.34) is satisfied trivially. If $k \neq 0$, one must also verify that (7.34) is satisfied by the selected solution of (7.35).

8. Examples. Homogeneous Flows:

To construct a simple wave $k$ ($\mu$) may be chosen arbitrarily. To construct double or triple waves it must be chosen to satisfy (6.7) or the first two equations of (7.8). In all cases $k = 0$ is a possible solution, and then by (4.7) the corresponding $\mu_\alpha$ are homogeneous of degree zero in the $x_I$. This suggests considering flows, not necessarily isentropic, in which $u_i$, $p$, $\rho$, and $s$ are homogeneous of degree zero in the $x_I$. If all of these functions are independent of $t$, a steady conical flow field is obtained. If some of these functions actually depend on $t$, let (8.1) $X_i = x_i/t$. Then $u_i$, $p$, $\rho$, and $s$ may be taken to be functions of $X_i$, and the geometry of the flow field simply expands at a uniform rate with the passage of time. Let $\nabla^* = \partial/\partial X_1, \partial/\partial X_2, \partial/\partial X_3$. Then (2.1 v) to (2.4 v) become

$$(8.2) \quad (\bar{u} - \bar{X}) . \nabla^* \bar{u} = -\rho^{-1} \nabla^* p$$

$$(8.3) \quad (\bar{u} - \bar{X}) . \nabla^* \rho + \rho \nabla^* . \bar{u} = 0$$

$$(8.4) \quad (\bar{u} - \bar{X}) . \nabla s = 0$$

In a homogeneous flow it is natural to seek characteristic hypersurfaces, defined by $f(X) =$ constant. These are conical hypersurfaces with vertices at the origin of space-time. Now the characteristic conditions take the form

$$(8.5) \quad (\bar{u} - \bar{X}) . \nabla^* f = 0$$

$$(8.6) \quad \left[ (\bar{u} - \bar{X}) . \nabla^* f \right]^2 = a^2 (\nabla^* f)^2$$

Note that if $\bar{u}$ is too close to $\bar{X}$, for example $\bar{u} = \bar{X}$, there are no solutions of (8.6), i.e. one cannot construct a conical characteristic hypersurface of the sound-wave type.

Three classical one dimensional examples of degenerate unsteady flows come immediately to mind. First, the early stages flow in a shock tube. Second, the very closely associated flows produced when a piston impulsively starts to advance into or recede from stagnant gas at constant speed. Third, the early stages of Lagrange's problem in interior ballistics, in which a mass of gas initially in a uniform state of rest propels a projectile down a tube without friction. These suggest the following generalizations to homogeneous unsteady flows.

1. Flows about cones in a shock tube of infinite cross-section.

Consider flows produced when an infinite plane shock moving with uniform velocity into stagnant gas and followed by an infinite region of uniform flow encounters a conical obstacle. Such flow include:

A. Development of Prandtl-Meyer flow around a corner (Fig. 1A 6, 7, 8). This would also arise in the early stages of the flow out of a plane shock tube with a flared exit (Fig. 1B).

B. Development of steady conical flow about cones not inclined too much relative to the flow (Fig. 2) [6]

C. Shock diffraction about and reflection from conical obstacles at large inclination to the flow (Fig. 3) [1, 4]

2. Conical Piston Problems.

These involve the flow about a conical object impulsively started into uniform translatory motion advancing into or receding from stagnant gas.

3. Conical Boundaries in Relative Motion.

An example of this type is an idealized model of an emerging projectile. Consider a plane block sliding along a fixed wall with a corner, and eventually separating from the corner. Assume that behind the block there is a region of uniform flow at high pressure, while between the block and the wall is stagnant gas at low pressure. Consider the homogeneous flow field after the block separates from the wall (Fig. 4).

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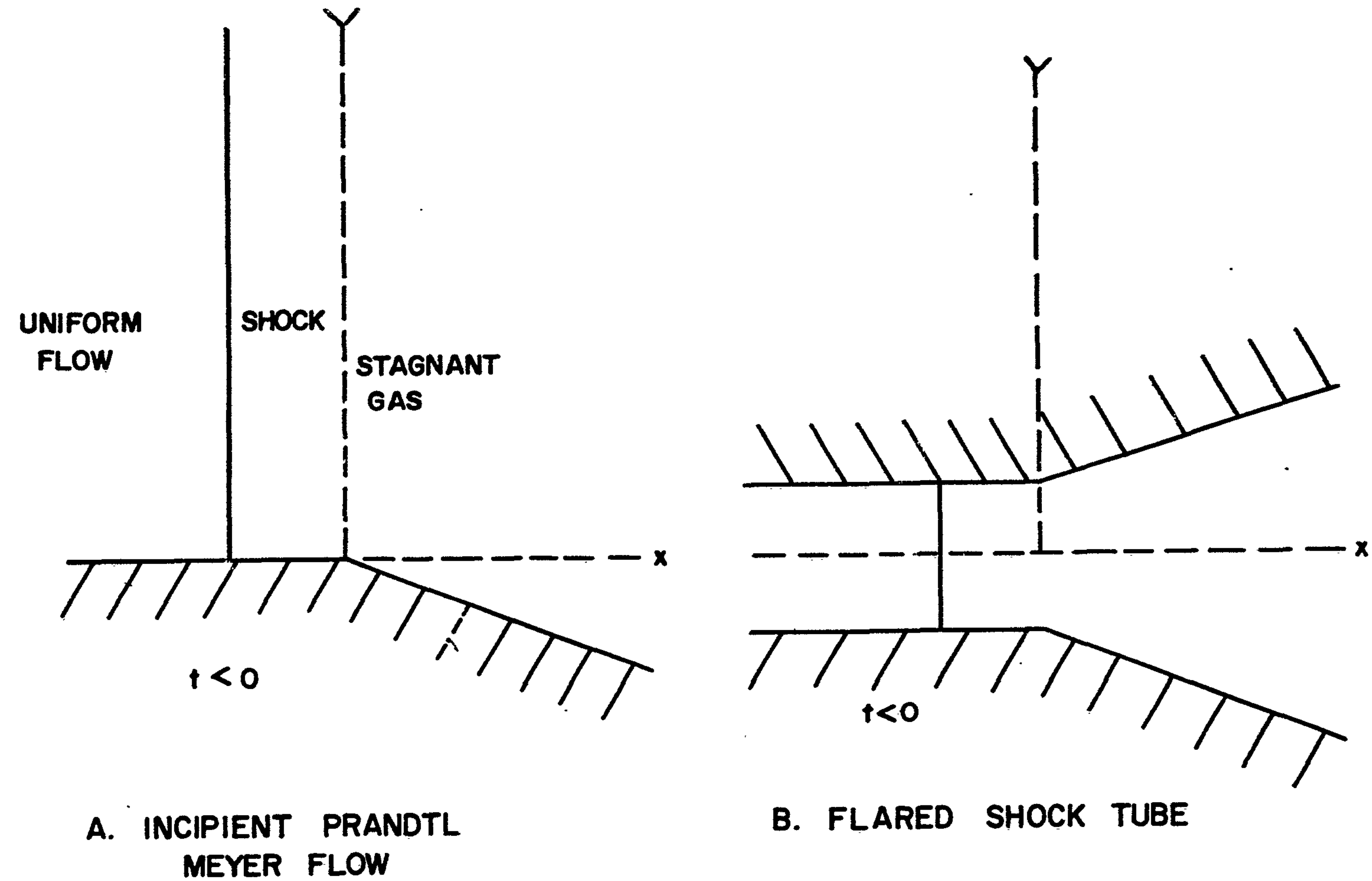

UNIFORM
FLOW
SHOCK
STAGNANT
GAS
Y
x
t < 0
A. INCIPIENT PRANDTL
MEYER FLOW
Y
x
t<0
B. FLARED SHOCK TUBE

FIGURE I

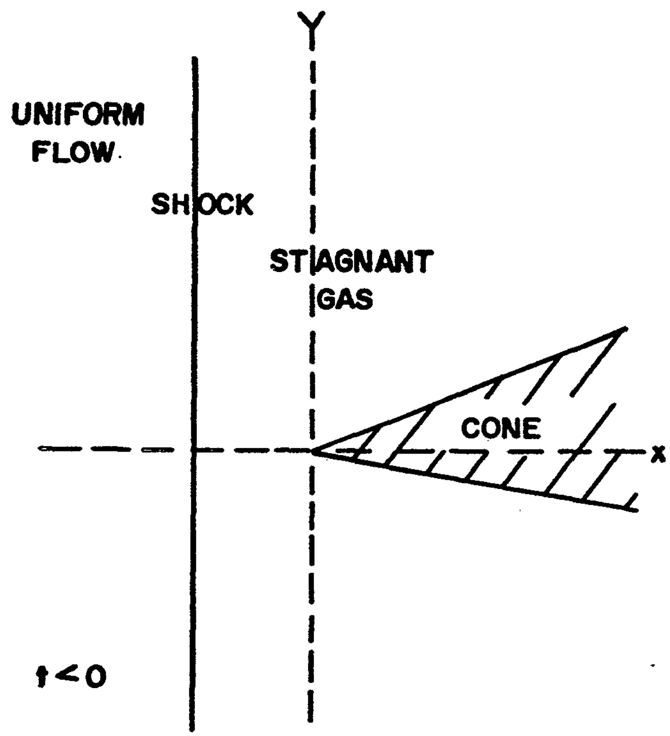



FIGURE 2 – INCIPIENT STEADY CONICAL FLOW

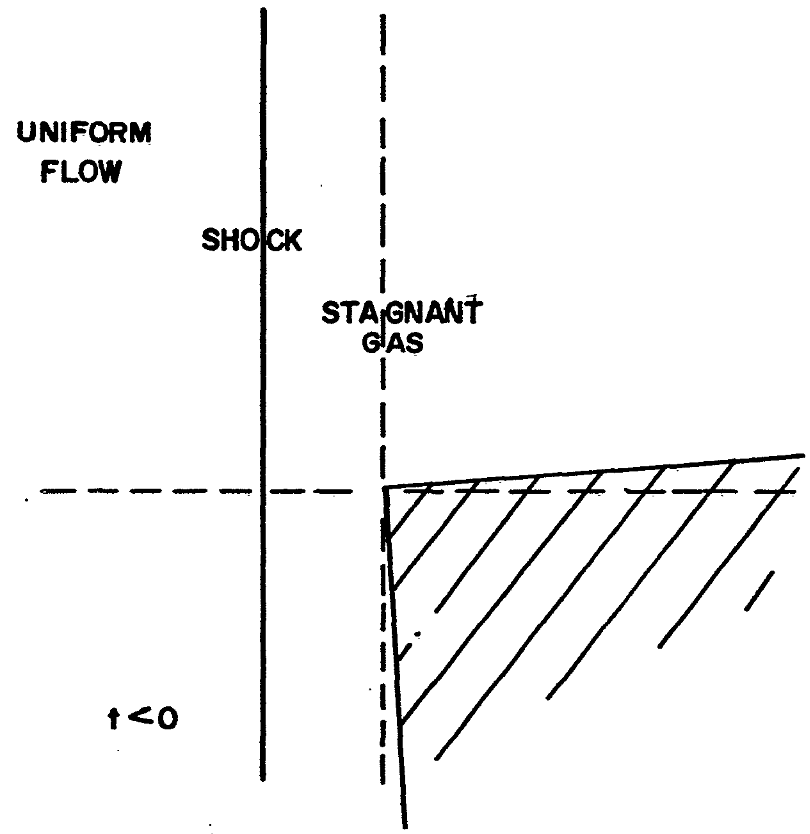
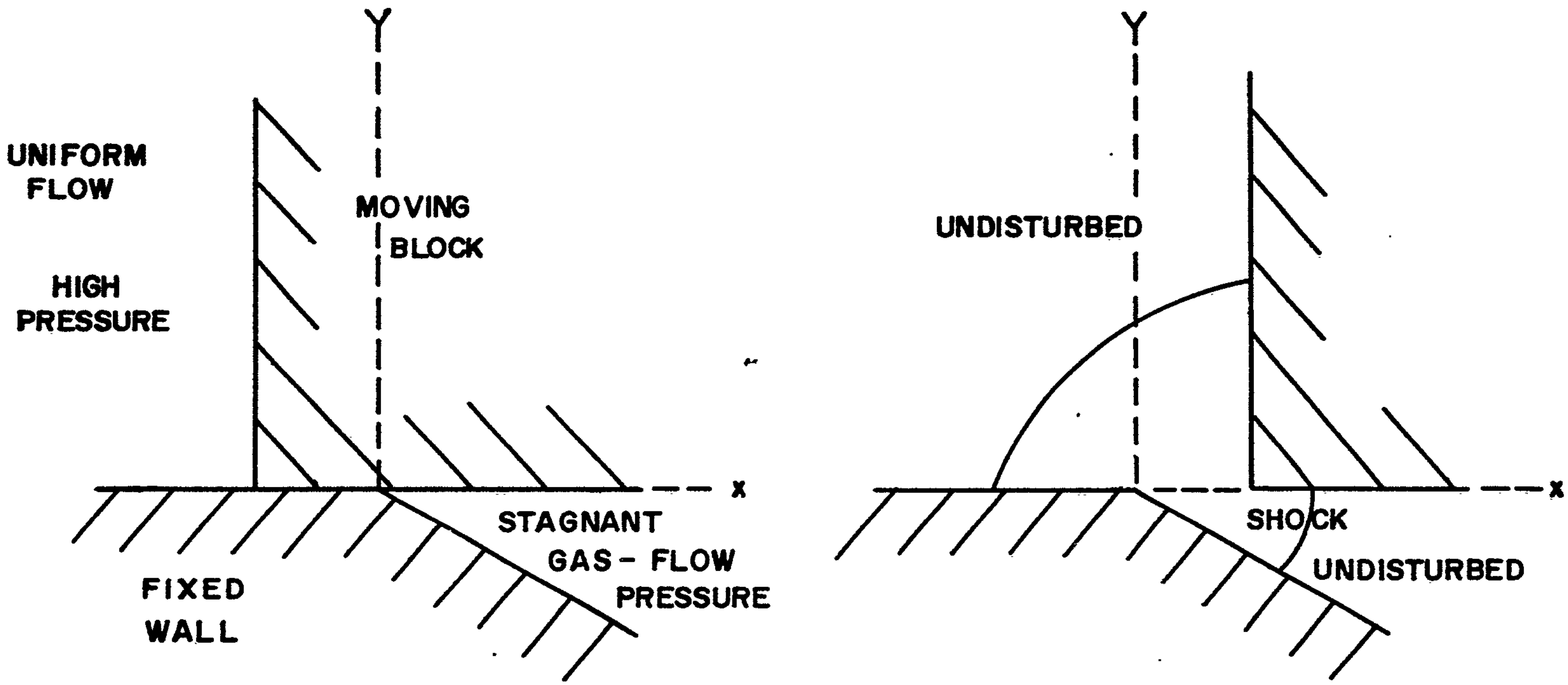



FIGURE 3 – SHOCK REFLECTION




A - T < 0 B — t > 0

FIGURE 4 - EMERGING PROJECTILE

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